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Pauli Equation for Non-Abelian Dyons in Moduli Space

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Undertaking the study of behaviour of an extended dyon moving in the generalized electromagnetic field of another non-Abelian dyon in moduli space, Dirac's equation has been solved for energy eigenvalues and it has been shown that spin momentum of an interacting non-Abelian dyon behaves as extra energy source. Introducing suitable spinors, the Pauli equation for a spin-1/2 non-Abelian dyon moving in the field of another non-Abelian dyon has been solved in moduli space and it has been shown that ad hoc introduction of spin in the system of two non-Abelian dyons perceptibly modifies the energy eigenvalues and eigenfunctions of bound states of the system.

1. INTRODUCTION

New interest in the subject of monopoles and dyons was enhanced by the work of ‘t Hooft (1974) and Polyakov (1974) and its extension by Julia and Zee (1975) and consequently these particles became intrinsic part of all current grand unified theories (Dokos and Tomaros, 1980; Preskill, 1984) with enormous potential importance (Callen, 1982a,b; Mandelstam, 1976a,b; Rajput, 1982, 1984; Rajput and Gunwant, 1988; Rubakov, 1981, 1982; ‘t Hooft, 1978; Witten, 1979). Keeping in view the results of Witten that monopoles are necessarily dyons, we have constructed a self-consistent and covariant quantum field theory of generalized electromagnetic field associated with dyons each carrying the generalized charge as complex quantity with electric and magnetic charges as its real imaginary parts (Rajput and Bhakuni, 1982; Rajput and Joshi, 1981). We have further undertaken the study of bound states and scattering of these particles with the help of this formalism and showed that bound states and scattering solutions are perceptibly modified due to the presence of magnetic charge of dyons (Pandey and Rajput, 1998, 1999; Pant *et al.*, 1997, 1999; Purohit *et al.*, 1999).

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Extending this work in the present paper, we have undertaken the study of Pauli equation for non-Abelian dyons in moduli space and showed that the interaction of spin and generalized potential leads to an extra-energy expressible in terms of generalized spin momentum of the particle concerned. Analyzing Dirac equation in moduli space, the study of interaction of spin and orbital angular momentum of this system has been undertaken. We have also undertaken the study of a spin-1/2 extended dyon moving in the field of another extended dyon by introducing suitable spinors and it has been demonstrated that bound state energy eigenvalues and eigenfunctions are perceptibly modifies from those of dyons in abelian as well as in non-Abelian gauge theories (Pandey and Rajput, 1998, 1999; Pant *et al.*, 1997, 1999; Purohit *et al.*, 1999).

2. BEHAVIOUR OF NON-ABELIAN DYON IN THE FIELD OF ANOTHER NON-ABELIAN DYON IN MODULI SPACE

Writing $\Psi(t, \vec{x})$ for a four-component spinor which also transforms under the fundamental representation of the SU(2) isospin group, the $(1 + 4)$-dimensional Dirac equation for a non-Abelian dyon in moduli space (Atiyah and Hitchin, 1988; Manton and Schroers, 1993) in the temporal gauge $V_0 = 0$, may be written as

$$[-\{\Gamma^0 \otimes \partial_t + c\Gamma^\mu \otimes D_\mu\} + mc^2]\Psi = 0 \tag{2.1}$$

where m is the mass of non-Abelian dyon and V_0 is the temporal part of the generalized four-potential $\vec{\vec{V}}_\mu = V_\mu^a T_a$. The vector sign $\rightsquigarrow$ denotes the internal group space; $\mu = 0, 1, 2, 3, 4$, represents degrees of freedom in the external space. The matrices $T_a (a = 1, 2, 3)$ are infinitesimal generators of the group SU(2) satisfying $[T_a, T_b] = \epsilon_{abc} T_c$, which can be expressed in terms of the Pauli matrices τ_a via $T_a = (1/2i)\tau_a$; dyonic generalized charge q with electric and magnetic constituents e and g is given by

$$q = e - ig \tag{2.2}$$

and similarly, the generalized four-potential is given as follows in terms of electric and magnetic four-potentials A_μ^a and B_μ^a, respectively:

$$V_\mu^a = A_\mu^a - iB_\mu^a. \tag{2.3}$$

Specifically, we consider SU(2) gauge potential $\vec{\vec{V}}_\mu, \mu = 1, 2, 3, 4$ on $R^4 = R^3 \times R$ which are independent of x_4. We can obtain five 4×4 complex matrices (Γ^0, Γ^μ), from the standard Dirac γ-matrices:

$$\Gamma^0 = \gamma^0, \quad \Gamma^i = \gamma^i, \quad \Gamma^4 = -i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.4}$$

Ψ really transforms under a spinor representation of SO(1,4) but we can think of it as an SO(1,3) spinor by restricting to the Lorentz transformations in SO(1,3) $\subset$ SO (1,4) respecting the condition $x_4 = 0$. Dirac's equation for non-Abelian dyon moving in an external field of another non-Abelian dyon in moduli space may then be written as

$$[-\{\Gamma^0 \otimes \partial_t + c\Gamma^\mu \otimes (\partial_\mu + |q|\,\vec{\vec{V}}_\mu)\} + mc^2]\Psi = 0. \tag{2.5}$$

Multiplying by Γ^4, we get

$$[-\Gamma^4\gamma^0 \otimes \partial_t - c\Gamma^4\gamma^i \otimes D_i - c(\Gamma^4)^2 \otimes (\partial_4 + |q|\,\vec{\vec{V}}_4) + \Gamma^4 mc^2]\Psi = 0. \tag{2.6}$$

Using $\Gamma^4 = 1_4$, $D_4 = \phi = c(\partial_4 + |q|\,\vec{\vec{V}}_4)$ and using Eq. (2.4), we get from Eq. (2.6);

$$\left[-\begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix} \otimes \partial_t - c\begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix} \otimes D_i - \begin{bmatrix} 1_2 & 0 \\ 0 & 1_2 \end{bmatrix} \otimes \phi \right.$$
$$\left. + \begin{bmatrix} 1_2 & 0 \\ 0 & 1_2 \end{bmatrix} mc^2 \right]\Psi = 0$$

or

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix}\Psi = \left[-c\begin{bmatrix} 0 & \sigma_i \otimes D_i \\ -\sigma_i \otimes D_i & 0 \end{bmatrix} - \begin{bmatrix} 1_2 \otimes \phi & 0 \\ 0 & 1_2 \otimes \phi \end{bmatrix}\right.$$
$$\left. + \begin{bmatrix} 1_2 & 0 \\ 0 & 1_2 \end{bmatrix} mc^2 \right]\Psi. \tag{2.7}$$

The relativistic energy of the particle includes also its rest energy mc^2. This must be excluded in arriving at the nonrelativistic approximation, and we therefore replace Ψ by a function Ψ' defined as follows:

$$\Psi = \Psi' e^{-imc^2 t/\hbar}.$$

Then from Eq. (2.7), we have

$$[i\hbar(\partial/\partial t) + mc^2]\begin{bmatrix} 1_2 & 0 \\ 0 & -1_2 \end{bmatrix}\Psi' = \left[-c\begin{bmatrix} 0 & \sigma_i \otimes D_i \\ -\sigma_i \otimes D_i & 0 \end{bmatrix}\right.$$
$$\left. - \begin{bmatrix} 1_2 \otimes \phi & 0 \\ 0 & 1_2 \otimes \phi \end{bmatrix} + \begin{bmatrix} 1_2 & 0 \\ 0 & 1_2 \end{bmatrix} mc^2 \right]\Psi'.$$

Substituting $\Psi' = [{}^{\xi}_{\eta}]$, where ξ and η are two-component functions. We obtain

$$\begin{bmatrix} [i\hbar(\partial/\partial t) + mc^2]\xi \\ [-\{i\hbar(\partial/\partial t) + mc^2\}]\eta \end{bmatrix} = \begin{bmatrix} -c\sigma_i \otimes D_i\eta \\ c\sigma_i \otimes D_i\xi \end{bmatrix} + \begin{bmatrix} -1_2 \otimes \phi\,\xi \\ -1_2 \otimes \phi\,\eta \end{bmatrix} + \begin{bmatrix} mc^2\,\xi \\ mc^2\,\eta \end{bmatrix}. \tag{2.8}$$

From Eq. (2.8), we get

$$[i\hbar(\partial/\partial t) + 1_2 \otimes \phi]\xi = -c\sigma_i \otimes (\partial_i + |q|\,\vec{\vec{V}}_i)\eta \tag{2.9}$$

$$-[i\hbar(\partial/\partial t) - 1_2 \otimes \phi + 2mc^2]\eta = c\sigma_i \otimes (\partial_i + |q|\,\vec{\vec{V}}_i)\xi. \tag{2.10}$$

In the first approximation, only the term $2mc^2\eta$ is retained on the left hand side of (2.10), which gives

$$\eta = -\frac{1}{2mc}\sigma_i \otimes (\partial_i + |q|\,\vec{\vec{V}}_i)\xi. \tag{2.11}$$

Substitution of (2.11) in (2.9), we get

$$i\hbar\frac{\partial\xi}{\partial t} = \left[\frac{1}{2m}(\vec{P}_i + |q|\,\vec{\vec{V}}_i)^2 - 1_2 \otimes \phi - \frac{|q|\hbar}{2m}\sigma_i \otimes \text{ curl } \vec{\vec{V}}_i\right]\xi = H\xi \tag{2.12}$$

where $\partial_i = \vec{P}_i$, is the momentum of non-Abelian dyon.

This is Pauli's equation for non-Abelian dyons in moduli space. It has the following extra spin contribution in the energy gained by spin-1/2 non-Abelian dyon while moving in the field of another non-Abelian dyon:

$$E' = -\frac{|q|\hbar}{2m}(\sigma_i \otimes \text{ curl } \vec{\vec{V}}_i). \tag{2.13}$$

This equation can also be written as

$$E' = -\mu_D \otimes \text{ curl } \vec{\vec{V}}_i = -\mu_{D'}(\sigma_i \otimes \text{ curl } \vec{\vec{V}}_i) \tag{2.14}$$

where

$$\mu_{D'} = \frac{|q|\hbar}{2m} \tag{2.15}$$

is defined as the Dyoneton for the system and

$$\mu_D = \mu_{D'}\sigma_i \tag{2.16}$$

as generalized spin moment of non-Abelian dyon. Consequently, extraenergy term in the Hamiltonian, may be interpreted as the energy of interaction of the generalized spin moment of non-Abelian dyon with a vector field, resulting from the space rotation of generalized four-potential. The third component of generalized spin moment operator for non-Abelian dyon may be written as

$$(\mu_D)_3 = \frac{|q|\hbar}{2m}\sigma_3, \tag{2.17}$$

the eigenvalue of which is

$$\pm\frac{|q|\hbar}{2m} = \pm\mu_{D'}. \tag{2.18}$$

3. SPIN–ORBIT INTERACTION

Let us consider the motion of a spin-1/2 non-Abelian dyon in the generalized electromagnetic field of another non-Abelian dyon retaining terms up to those of order v^2/c^2. Substituting $\vec{\vec{V}}_i = 0$ and $E = i\hbar(\partial/\partial t)$ in Eqs. (2.9) and (2.10), we find

$$(E + 1_2 \otimes \phi)\xi = -c\sigma_i \otimes \vec{P}_i\eta \tag{3.1}$$

$$-(E - 1_2 \otimes \phi + 2mc^2)\eta = c\sigma_i \otimes \vec{P}_i\xi. \tag{3.2}$$

We calculate from (3.2) the function η up to terms of first order in $(E - l_2 \otimes \phi)/2mc^2$. Substituting the value

$$\eta = -\frac{1}{2mc}\left[1 - \frac{E - 1_2 \otimes \phi}{2mc^2}\right](\sigma_i \otimes \vec{P}_i)\,\xi$$

into Eq. (3.1), we find an equation containing only one two-component function

$$(E + 1_2 \otimes \phi)\xi = \frac{1}{2m}(\sigma_i \otimes \vec{P}_i)\left[1 - \frac{E - 1_2 \otimes \phi}{2mc^2}\right](\sigma_i \otimes \vec{P}_i)\,\xi \tag{3.3}$$

which on simplification gives the following expression for energy operator (Hamiltonian) in the first approximation:

$$H = \frac{1}{2m}\left[1 - \frac{E - 1_2 \otimes \phi}{2mc^2}\right]\vec{P}_i^{\,2} - 1_2 \otimes \phi - [i\hbar/4m^2c^2][\vec{\nabla}(1_2 \otimes \phi) \otimes \vec{P}_i]$$
$$+ [\hbar/4m^2c^2][\sigma_i \otimes \{\vec{\nabla}(1_2 \otimes \phi) \times \vec{P}_i]. \tag{3.4}$$

In order to derive expression for Hamiltonian, in second approximation we use instead of ξ another function χ, given by

$$\chi = \hat{u}\xi$$

the normalization of which up to second order leads to the following value of factor u:

$$\hat{u} \approx 1 - \left[\vec{P}_i^{\,2}/8m^2c^2\right].$$

Using this value of $\hat{u}$ (and hence of χ), we get the following relativistic expression for corresponding Hamiltonian, up to terms of order v^2/c^2:

$$\begin{aligned}
\hat{H} &= \left[1 + \left(\vec{P}_i^{\,2}/8m^2c^2\right)\right]\hat{H}\left[1 - \left(\vec{P}_i^{\,2}/8m^2c^2\right)\right] \\
&= \left[\left(\vec{P}_i^{\,2}/2m\right) - 1_2 \otimes \phi\right] + [(\hbar^2/8m^2c^2)\vec{\nabla}^2(1_2 \otimes \phi)] - [(E - 1_2 \otimes \phi)^2/2mc^2] \\
&\quad + [(\hbar/4m^2c^2)[\sigma_i \otimes \{\vec{\nabla}(1_2 \otimes \phi) \times \vec{P}_i\}]] \\
&= \hat{H}_0 + \hat{H}_1 + \hat{H}_2 + \hat{H}_3 \\
&= \hat{H}_0 + \hat{H}_1
\end{aligned} \tag{3.5}$$

where $\hat{H}_0$ corresponds to the nonrelativistic term of the Hamiltonian, whereas $\hat{H}_1$ is the relativistic correction term to the Hamiltonian various parts of which arise due to different relativistic interaction. The quantity $\hat{H}_1$ is called contact interaction operator, analogous to the term introduced by Darwin (1928) for electronic case. $\hat{H}_2$ is the relativistic correction term due to the dependence of kinetic energy on momentum. Finally,

$$\hat{H}_3 = \frac{\hbar[\sigma_i \otimes \{\vec{\nabla}(l_2 \otimes \phi) \times \vec{P}_i\}]}{4m^2c^2} \tag{3.6}$$

is the so-called spin orbit interaction operator.

In a spherically symmetric field

$$\vec{\nabla}\phi = \frac{\vec{r}}{r}\frac{d\phi}{dr}.$$

Substituting this expression into (3.6) we find the spin–orbit interaction operator for the motion of a spin-1/2 particle in a spherically symmetric field:

$$\hat{H}_3 = \frac{d}{dr}(1_2 \otimes \phi)\frac{\hat{S} \otimes \hat{L}}{2m^2c^2r} \tag{3.7}$$

where $\hat{L} = \vec{r} \times \vec{P}_i$ is the orbital angular momentum operator and $\hat{S} = {}^1/_2\hbar\sigma_i$ is the spin angular momentum operator. This expression clearly demonstrates that besides the contribution of Higgs field, the interaction of spin and orbital angular momenta of moving non-Abelian dyon also contributes to the energy operator.

4. PAULI EQUATION FOR A NON-ABELIAN DYON IN THE FIELD OF ANOTHER NON-ABELIAN DYON IN MODULI SPACE

For analyzing the motion of spin-1/2 non-Abelian dyon in the field of another non-Abelian dyon with the inclusion of spin effect, let us start with the following Schrodinger equation for *i*th dyon moving in the field of *j*th dyon in non-Abelian gauge form in moduli space, may be written as

$$\left[-\frac{1}{2m}\hat{\nabla}^2 + 1_4 \otimes \phi(r) + F(r)\hat{L} \otimes \sigma_i\right]\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} = E\begin{bmatrix}\Psi_1\\ \Psi_2\end{bmatrix} \tag{4.1}$$

where $\hat{\nabla}$ is $(\vec{P} + |q|\,\vec{\vec{V}})$, $\phi(r)$ is the Higgs potential; through a specific kind of gauge transformation which sends the operator $\hat{r} \cdot \vec{T}$ to an operator $\vec{T}_3$ gives the following form of Higgs potential:

$$\phi(r) = -\frac{\alpha_{ij}}{r} + \frac{\mu_{ij}^2}{2mr^2} \tag{4.2}$$

where α_{ij} is electric coupling parameter and μ_{ij} is the magnetic coupling parameter. And spin–orbit interaction $F(r)\hat{L} \otimes \sigma_i$ will be treated as small perturbation. Though the nonrelativistic Pauli equation (4.1) is not sufficiently complex to yield precise values for fine structure of dyonium energy level, it can be safely taken as a useful guide to an understanding the role of spin in bound states of two dyons, i.e. dyonium.

The unperturbed Hamiltonian

$$\hat{H}_0 = (-1/2m)\hat{\nabla}^2 + 1_4 \otimes \phi(r), \tag{4.3}$$

represents a central force problem for dyonium and the spin–orbit interaction energy $\hat{H}'$ is given by

$$\hat{H}' = \frac{\alpha_{ij}}{2m^2c^2}\left\langle\frac{1}{r^3}\right\rangle 1_2 \otimes \hat{S} \otimes \hat{L} - \frac{\mu_{ij}^2}{2m^3c^2}\left\langle\frac{1}{r^4}\right\rangle 1_2 \otimes \hat{S} \otimes \hat{L} \tag{4.4}$$

where the symbols have their usual meaning. To simplify the above equation, we introduce the total angular momentum as

$$\hat{J} \otimes \hat{J} = \hat{L} \otimes \hat{L} + \hat{S} \otimes \hat{S} + 2\,\hat{L} \otimes \hat{S}.$$

So the Pauli operator for $\hat{H}'$ is given by

$$\begin{aligned}(\hat{H}')_P = {} & \frac{\alpha_{ij}}{4m^2c^2}\left\langle\frac{1}{r^3}\right\rangle 1_2 \otimes [(\hat{J} \otimes \hat{J})_P - (\hat{L} \otimes \hat{L})_P - (\hat{S} \otimes \hat{S})_P] \\ & - \frac{\mu_{ij}^2}{4m^3c^2}\left\langle\frac{1}{r^4}\right\rangle 1_2 \otimes [(\hat{J} \otimes \hat{J})_P - (\hat{L} \otimes \hat{L})_P - (\hat{S} \otimes \hat{S})_P].\end{aligned} \tag{4.5}$$

Thus, the Pauli wave equation becomes

$$(\hat{H})_{P\Psi_P} = [(\hat{H}_0)_P + (\hat{H}')_P]_{\Psi_P} = W_{\Psi_P}, \tag{4.6}$$

where

$$(\hat{H}_0)_p = \begin{bmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix}_P = \begin{bmatrix} -(1/2m)\hat{\nabla}^2 - \alpha_{ij}1_4/r + \mu_{ij}^2 1_4/2mr^2 & 0 \\ 0 & -(1/2m)\hat{\nabla}^2 - \alpha_{ij}1_4/r + \mu_{ij}^2 1_4/2mr^2 \end{bmatrix} \tag{4.7}$$

and

$$\Psi_P = \begin{bmatrix} \Psi_+ \\ \Psi_- \end{bmatrix}_P \tag{4.8}$$

represents the Pauli wave function. The Pauli wave equation for unperturbed Hamiltonian is given as

$$\begin{bmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_0 \end{bmatrix}\begin{bmatrix} \Psi_+(0) \\ \Psi_-(0) \end{bmatrix} = W(0)\begin{bmatrix} \Psi_+(0) \\ \Psi_-(0) \end{bmatrix} \tag{4.9}$$

or

$$\hat{H}_{0\Psi\pm}(0) = W(0)_{\Psi\pm}(0). \tag{4.10}$$

This wave equation is solved by introducing the total angular momentum operator as vector sum of the orbital angular momentum in the gauge field, isotopic spin, and spin

$$\vec{J} = \vec{L} + \vec{S},$$
$$\vec{L} = \vec{M} + (\vec{T} \cdot \hat{r})r,$$
$$\vec{M} = \vec{r} \times \left[P - \frac{\vec{r} \times \vec{T}}{r} \right], \tag{4.11}$$

which satisfy the following eigenvalue equations for the angular momentum eigenfunction:

$$\begin{bmatrix} J^2 \\ L^2 \\ J_3 \\ T^2 \end{bmatrix} Y_{\mu ij,l,m}(\theta, \phi) = \begin{bmatrix} j(j+1) \\ l(l+1) \\ m_j \\ t(t+1) \end{bmatrix} Y_{\mu ij,l,m}(\theta, \phi), \tag{4.12}$$

where $Y_{\mu ij,l,m}(\theta, \phi)$ are the dyon harmonics (Pandey *et al.*, 1990) and the radial function $[U(r)/r] = R(r)$ satisfy the equation

$$\Psi = \frac{U(r)}{r} Y_{\mu ij,l,m}(\theta, \phi). \tag{4.13}$$

We get the following radial equation after separation of variables by substituting Eq. (4.13) into Eq. (4.10),

$$r^2 \left\{ \frac{1}{rR(r)} \frac{d^2}{dr^2}(rR) + 2m(E - 1_4 \otimes \phi) \right\} = -\frac{\Lambda Y_{\mu ij,l,m}(\theta, \phi)}{Y_{\mu ij,l,m}(\theta, \phi)} = l(l+1) \tag{4.14}$$

with

$$\Lambda = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left[\sin\theta \frac{\partial}{\partial\theta} \right] + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \tag{4.15}$$

and

$$\phi = -\frac{\mathrm{Re}(q_i q_j^*)}{r} + \frac{[\mathrm{Im}(q_i q_j^*)]^2}{2mr^2}. \tag{4.16}$$

Substituting the value of ϕ from (4.16) into (4.14), we get

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{dR}{dr} \right] + 2m \left[E + \frac{\mathrm{Re}(q_i q_j^*) 1_4}{r} - \frac{[\mathrm{Im}(q_i q_j^*)]^2 1_4}{2mr^2} - \frac{l(l+1)}{2mr^2} \right] R(r) = 0. \tag{4.16a}$$

Substituting dimensionless variable $\rho = \alpha r$ this equation becomes

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2\frac{dR}{d\rho}\right] + \left[\frac{2mE}{\alpha^2} + \frac{2m\,\mathrm{Re}(q_i q_j^*)1_4}{\alpha\rho} - \frac{l(l+1) - [\mathrm{Im}(q_i q_j^*)]^2 1_4}{2m\alpha^2\rho^2}\right]R(\rho) = 0, \qquad (4.16b)$$

where $\alpha^2 = 8m|E| = -8mE$.

Equation (4.16b) may also be written as

$$\frac{1}{\rho^2}\frac{d}{d\rho}\left[\rho^2\frac{dR}{d\rho}\right] + \left[\frac{\lambda}{\rho} - \frac{1}{4} - \frac{S(S+1)}{\rho^2}\right]R(\rho) = 0, \qquad (4.17)$$

where

$$S(S+1) = \frac{2ml(l+1) - [\mathrm{Im}(q_i q_j^*)]^2 1_4\sqrt{}(m/-2E)}{2m} \qquad (4.18)$$

and

$$\lambda = \mathrm{Re}(q_i q_j^*)1_4\sqrt{}(m/-2E).$$

Equation (4.17) yields the following energy eigenvalue for the system of an extended dyon spinning around another non-Abelian dyon:

$$E_n = -2m[\mathrm{Re}(q_i q_j^*)]^2 1_4[(2n+1) + \{(21+1)^2 + (\mathrm{Im}(q_i q_j^*)/m)^2 1_4\}^{1/2}]^{-2}, \qquad (4.19)$$

where $n = 0, 1, 2, \ldots$ and $\Psi_\pm(0)$ are dyonium wave functions simplified to $R_{nl}(r)$ $Y_{\mu ij,l,m}(\theta, \phi)$, where $Y_{\mu ij,l,m}(\theta, \phi)$ are dyon harmonics (Pandey *et al.*, 1990). Thus, the Pauli wave function for spin up and spin down states are given by

$$[\Psi_+(0)]_P = \Psi_{(n,\,l,\,ml,\,ms=+1/2)} = R_{nl}Y_{\mu ij,m,l}|\uparrow\rangle = \begin{bmatrix} R_{nl}Y_{\mu ij,m,l} \\ 0 \end{bmatrix} \qquad (4.20)$$

and

$$[\Psi_-(0)]_p = \Psi_{(n,l,ml,ms=-1/2)} = R_{nl}Y_{\mu ij,m,l}|\downarrow\rangle = \begin{bmatrix} 0 \\ R_{nl}Y_{\mu ij,l,m} \end{bmatrix}. \qquad (4.21)$$

In the absence of spin–orbit interaction, both the wave functions corresponds to the same energy. In order to determine the splitting due to spin–orbit interaction we should choose a representation in which H' is diagonal:

$$(\phi_1)_P = \phi_{(n,\,l,\,j=l+1/2,\,mj)}$$

or

$$(\phi_1)_p = \sqrt{\frac{l+m_j+1/2}{2l+1}}\Psi_{(n,\,l,\,ml=mj-1/2, ms=+1/2)} + \sqrt{\frac{l-m_j+1/2}{2l+1}}\Psi_{(n,\,l,\,ml=mj+1/2,\,ms=-1/2)} = \begin{bmatrix} \sqrt{\frac{l+m_j+1/2}{2l+1}} R_{nl} Y_{\mu ij,l,mj-1/2} \\ \sqrt{\frac{l-m_j+1/2}{2l+1}} R_{nl} Y_{\mu ij,l,mj+1/2} \end{bmatrix} \tag{4.22}$$

Similarly, we can write $(\phi_2)_p = \phi_{(n,l,j=1-1/2,mj)}$.

Then the first order perturbation due to the spin–orbit interaction would be given by

$$\begin{aligned} W_s^{(1)} &= \int d\tau \phi^\dagger (H')_P \phi \\ &= (\mathrm{Re}(q_i q_j^*)/4m^2c^2) \int d\tau (1/r^3) \phi^\dagger 1_2 \otimes [(\hat{J}\otimes\hat{J})_P - (\hat{L}\otimes\hat{L})_P \\ &\quad - (\hat{S}\otimes\hat{S})_P]\phi - (\mathrm{Im}(q_i q_j^*)^2/4m^3c^2) \\ &\quad \times \int d\tau (1/r^4)\phi^\dagger 1_2 \otimes [(\hat{J}\otimes\hat{J})_P - (\hat{L}\otimes\hat{L})_P - (\hat{S}\otimes\hat{S})_P]\phi \end{aligned} \tag{4.23}$$

or

$$\begin{aligned} W_s^{(1)} &= (\mathrm{Re}(q_i q_j^*)/4m^2c^2)[1_2 \otimes \{j(j+1) - l(l+1) - 3/4\}] \\ &\quad \times \int d\tau (1/r^3)[\{(l \pm m_j + 1/2)/(2l+1)\}|R_{nl}|^2 |Y_{\mu ij,l,mj-1/2}|^2 \\ &\quad + \{(l \mp m_j + 1/2)/(2l+1)\}|R_{nl}|^2 |Y_{\mu ij,l,mj+1/2}|^2] \\ &\quad + (\mathrm{Im}(q_i q_j^*)^2/4m^3c^2)[1_2 \otimes \{j(j+1) - l(l+1) - 3/4\}] \\ &\quad \times \int d\tau (1/r^4)[\{(l \pm m_j + 1/2)/(2l+1)\}|R_{nl}|^2 |Y_{\mu ij,l,mj-1/2}|^2 \\ &\quad + \{(l \mp m_j + 1/2)/(2l+1)\}|R_{nl}|^2 |Y_{\mu ij,l,mj+1/2}|^2], \end{aligned} \tag{4.24}$$

where the upper and lower signs corresponds to $j = l + 1/2$ and $j = l - 1/2$, respectively. After integration we get

$$W_s^{(1)} = \begin{cases} \dfrac{\mathrm{Re}\,(q_i q_j^*)}{4m^2c^2} 1_2 \otimes l \left\langle \dfrac{1}{r^3} \right\rangle + \dfrac{\mathrm{Im}\,(q_i q_j^*)^2}{4m^3c^2} 1_2 \otimes l \left\langle \dfrac{1}{r^4} \right\rangle \quad \text{for} \quad j = l + \dfrac{1}{2}, \\ -\dfrac{\mathrm{Re}\,(q_i q_j^*)}{4m^2c^2} 1_2 \otimes (l+1) \left\langle \dfrac{1}{r^3} \right\rangle + \dfrac{\mathrm{Im}\,(q_i q_j^*)^2}{4m^3c^2} 1_2 \otimes (l+1) \left\langle \dfrac{1}{r^4} \right\rangle \\ \quad \text{for} \quad j = l - \dfrac{1}{2}, \end{cases} \tag{4.25}$$

where

$$\left\langle \frac{1}{r^3} \right\rangle = \int_0^\infty \frac{1}{r^3} |R_{nl}|^2 r^2\, dr = \frac{1}{n^3 l(l+1/2)(l+1)} \frac{1}{a_0^2},$$
$$\left\langle \frac{1}{r^4} \right\rangle = \int_0^\infty \frac{1}{r^4} |R_{nl}|^2 r^2\, dr = \frac{3 - 5n^3(l+1/2)a_0^2}{n^5(l-1/2)(l+1/2)(l+3/2)a_0^4}. \tag{4.26}$$

The splitting of energy levels corresponding to quantum number n is $W = W^{(0)} + W_s^{(1)}$

$$W = \begin{cases} E_n - \dfrac{E_n\,\mathrm{Re}(q_i q_j^*) 1_2 \otimes l}{2m^2c^2n^3 l(l+1)(2l+1)a_0^2} \\ \quad - \dfrac{E_n\,\mathrm{Im}(q_i q_j^*)^2 1_2 \otimes l[3 - 5n^3(l+1/2)a_0^2]}{m^4c^2n^5(2l-1)(2l+1)(2l+3)a_0^4} \quad \text{for} \quad j = l + \dfrac{1}{2}, \\ E_n + \dfrac{E_n\,\mathrm{Re}(q_i q_j^*) 1_2 \otimes (l+1)}{2m^2c^2n^3 l(l+1)(2l+1)a_0^2} \\ \quad - \dfrac{E_n\,\mathrm{Im}(q_i q_j^*)^2 1_2 \otimes (l+1)[3 - 5n^3(l+1/2)a_0^2]}{m^4c^2n^5(2l-1)(2l+1)(2l+3)a_0^4} \quad \text{for} \quad j = l - \dfrac{1}{2} \end{cases} \tag{4.27}$$

where E_n is given by Eq. (4.19) and the Bohr radius a_0 for this system is given as

$$a_0 = \frac{[\mathrm{Im}(q_i q_j^*)]^2 + 1}{m\,\mathrm{Re}(q_i q_j^*)}. \tag{4.28}$$

Equation (4.27) gives the splitting in the energy levels corresponding to quantum number n for $j = l + 1/2$ and $j = l - 1/2$, respectively.

5. CONCLUSION

Equation (2.5) is Dirac's equation for extended dyon moving in generalized electromagnetic field of another non-Abelian dyon in moduli space, which on solving gives Pauli's equation (2.12) for non-Abelian dyons. Equation (3.5) is the relativistic Hamiltonian, for non-Abelian dyon in field of another non-Abelian dyon in moduli space, different parts of which arise due to different relativistic interactions. Hamiltonian (3.5) of this system has been shown in terms of Higgs potential instead of scalar potential in abelian as well as in non-Abelian gauge theories (Rajput and Pandey, 1998) due to moduli space approximation. Equation (4.1) is the Schrodinger equation for a spinning non-Abelian dyon in the field of another non-Abelian dyon in moduli space, in this equation spin has been introduced in an ad hoc manner. Equation (4.19) described the energy eigenvalue of this system and Eq. (4.20) and (4.21) describe the Pauli wave function associated with spinning non-Abelian dyons in moduli space. Equation (4.27) is the splitting of energy levels corresponding to quantum number *n* and Eq. (4.28) is Bohr radius for non-Abelian dyonium in moduli space.

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